

# A VORTICITY-MAGNETIC FIELD DYNAMO INSTABILITY

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## ABSTRACT

We generalize the mean field magnetic dynamo to include local evolution of the mean vorticity in addition to the mean magnetic field. The coupled equations exhibit a general mean field dynamo instability that enables the transfer of turbulent energy to the magnetic field and vorticity on larger scales. The growth of the vorticity and magnetic field both require helical turbulence which can be supplied by an underlying global rotation. The dynamo coefficients are derived including the backreaction from the mean magnetic field to lowest order. We find that a mean vorticity field can actually seed exponential growth of mean magnetic field from only a small scale seed magnetic field, without the need for a seed mean magnetic field. The equations decouple when the fluctuating velocity and magnetic field cross-correlations vanish, resulting in the separate mean field magnetic and mean field vorticity dynamo equations.

Subject Headings: magnetic fields: MHD; instabilities; accretion disks; sun: magnetic fields; galaxies: jets.

## 1. Introduction

The mean-field magnetic dynamo (MFMD) has been a standard framework for understanding the origin of large scale magnetic fields (B-fields) in planets stars and galaxies (Moffatt 1978, Parker 1979, Ruzmaikin et al. 1988, Beck et al. 1996). In this mechanism, a large scale seed field grows exponentially at the expense of small scale turbulent energy. This can be distinguished from fast dynamo turbulent amplification of B-fields (e.g. Parker 1979), which occurs on the scale of the input turbulence. Many systems such as planetary and solar spots (Peter 1996), and accretion and galactic disks (Abramowicz 1992, Kitchatinov et al. 1994a) also show evidence for large scale vortex structures. Jets may also be related. In the absence of B-fields, the mean vorticity equation is similar to the mean magnetic induction equation (Kitchatinov et al. 1994a) and can allow mean field vorticity dynamo (MFVD) growth. When the mutual coupling of vorticity and B-field is considered, the mean quantity whose time evolution is of interest has 6 components: 3 components each for the mean B-field and mean vorticity. Growth arises in this coupled mean field dynamo (CMFD).

We first derive the equations for mean and fluctuating quantities and show that the evolution equations for the mean vorticity and B-field are non-trivial only when the turbulence is at least weakly inhomogeneous and anisotropic. By expanding the turbulent quantities to first order in the time-varying mean velocity and mean B-field, we derive the initial mutual dynamo growth for several sets of initial conditions. When cross-correlations between turbulent velocities and turbulent B-fields vanish, mutual field growth is described by decoupled MFVD and MFMD equations. However, when the cross correlations do not vanish, the time evolution of the mean fields is coupled. As a result, we find that a seed mean vorticity and small scale B-field can allow growth of the mean B-field even when the latter is ini-

tially zero, in contrast to the decoupled theory.

We include the mean B-field backreaction on the velocity flows to lowest order, thus deriving corrections to the kinematic magnetic and vorticity dynamo coefficients. However, we do not compute these corrections to higher order, nor do we purport to offer a fully developed non-linear theory of MHD turbulence. We make explicit assumptions/approximations, but emphasize that none of these are beyond those used in standard mean-field dynamo treatments. (In many treatments, the same approximations are imposed without explanation.) We are aware of the controversies of the non-linear backreaction (e.g. Piddington 1981, Kulsrud & Anderson 1982, Field & Blackman, 1996, Cattaneo 1994, Pouquet et al. 1976, Blackman 1996, Brandenburg 1996, Subramanian 1997), but inasmuch as the “ $\alpha^2$ ” (e.g. Moffatt 1978) or “ $\alpha - \Omega$ ” (e.g. Parker 1979) dynamos are at least a framework for the study of B-field generation and comparison with observations, so too is the more generalized treatment of the vorticity-magnetic field dynamo developed herein.

## 2. Derivation of Coupled Mean-Field Equations

Consider the Navier-Stokes equation for incompressible flows in the presence of B-fields,

$$\begin{aligned} \partial_t \mathbf{v} = & \mathbf{v} \times (\nabla \times \mathbf{v}) - \nabla p_{eff} + \nu \nabla^2 \mathbf{v} + \mathbf{b} \cdot \nabla \mathbf{b} \\ & + \mathbf{F}(\mathbf{x}, t) + \nabla \phi, \end{aligned} \quad (1)$$

where  $\nu$  is a constant viscosity,  $\mathbf{F}$  is a forcing function,  $p_{eff} \equiv p + b^2/2 + v^2/2$ , with  $p \equiv P/\rho$  and  $\mathbf{b} \equiv \mathbf{B}/(4\pi\rho)^{1/2}$  is the constant density normalized B-field. The  $\nabla\phi$  includes potential forces such as gravity. The vorticity equation,

$$\partial_t \boldsymbol{\omega} = \nabla \times (\mathbf{v} \times \boldsymbol{\omega}) + \nu \nabla^2 \boldsymbol{\omega} + \nabla \times (\mathbf{b} \cdot \nabla \mathbf{b}) + \nabla \times \mathbf{F}(\mathbf{x}, t), \quad (2)$$

where  $\boldsymbol{\omega} \equiv \nabla \times \mathbf{v}$ , is obtained by taking the curl of (1). The induction equation for  $\mathbf{b}$  is

$$\partial_t \mathbf{b} = \nabla \times (\mathbf{v} \times \mathbf{b}) + \nu_M \nabla^2 \mathbf{b}, \quad (3)$$

where  $\nu_M$  is a constant magnetic viscosity.

Next, decompose  $\boldsymbol{\omega}$ ,  $\mathbf{v}$ , and  $\mathbf{b}$  into mean (indicated by an overbar or  $\langle \rangle$ ) and fluctuating (indicated by prime) components  $\mathbf{v} = \bar{\mathbf{v}}_T + \mathbf{v}'$ ,  $\boldsymbol{\omega} = \bar{\boldsymbol{\omega}}_T + \boldsymbol{\omega}'$  and  $\mathbf{b} = \bar{\mathbf{b}} + \mathbf{b}'$ , respectively with  $\bar{\mathbf{v}}_T = \bar{\mathbf{v}} + \bar{\mathbf{v}}_c$ . Here,  $\bar{\mathbf{v}}_c$  is a constant global velocity whose curl is  $\bar{\boldsymbol{\omega}}_c$ , and  $\bar{\mathbf{v}}$  and  $\bar{\boldsymbol{\omega}}$  are the time dependent mean components whose spatial scales of variation are smaller spatial than that of  $\bar{\boldsymbol{\omega}}_c$ . The  $\bar{\boldsymbol{\omega}}_c$  supplies helicity to the turbulence and ensures that any growth of  $\bar{\boldsymbol{\omega}}$  represents a transfer of angular momentum from the turbulence. (Alternatively, the helicity could be supplied by  $\mathbf{F}(\mathbf{x}, t)$ .)

We assume that derivatives with respect to  $\mathbf{x}$  or  $t$  obey  $\partial_{t,\mathbf{x}} \langle X_i X_j \rangle = \langle \partial_{t,\mathbf{x}} (X_i X_j) \rangle$  and  $\langle \bar{X}_i X'_j \rangle = 0$  (Reynolds relations (Rädler 1980)), where  $X_i = \bar{X}_i + X'_i$  are components of vector functions of  $\mathbf{x}$  and  $t$ . For statistical ensemble means, these hold when the correlation time scales are short relative to the variation times of mean quantities. For the spatial mean, defined by  $\langle X_i(\mathbf{x}, t) \rangle = |\zeta|^{-3} \int_{\mathbf{x}-L}^{\mathbf{x}+L} X_i(\mathbf{s}, t) d^3 \mathbf{s}$ , the relations hold when the averaging is taken over a large enough scale, such that  $l \ll |\zeta| \ll L$ , where  $L \sim \bar{b}/\nabla \bar{b}$ , and  $\ell \sim b'/\nabla b' \sim v'/\nabla v'$ .

Subtracting the mean of (1) from itself, and assuming  $\nabla \phi' = 0$ , gives

$$\begin{aligned} d_t \mathbf{v}' = & \langle \mathbf{v}' \cdot \nabla \mathbf{v}' \rangle - \mathbf{v}' \cdot \nabla \mathbf{v}' - \bar{\mathbf{v}} \cdot \nabla \mathbf{v}' - \mathbf{v}' \cdot \nabla \bar{\mathbf{v}}_T - \\ & \nabla p' - \nabla b'^2/2 + \nabla \langle b'^2 \rangle/2 - \nabla (\mathbf{b}' \cdot \bar{\mathbf{b}}) + \mathbf{b}' \cdot \nabla \bar{\mathbf{b}} - \\ & \langle \mathbf{b}' \cdot \nabla \mathbf{b}' \rangle + \mathbf{b}' \cdot \nabla \bar{\mathbf{b}} + \bar{\mathbf{b}} \cdot \nabla \mathbf{b}' + \mathbf{F}'(\mathbf{x}, t) + \nu \nabla^2 \mathbf{v}', \end{aligned} \quad (4)$$

where  $d_t \equiv \partial_t + \bar{\mathbf{v}}_c \cdot \nabla$ . The average of (2) is

$$\begin{aligned} d_t \bar{\boldsymbol{\omega}} = & \nabla \times \langle \mathbf{v}' \times \boldsymbol{\omega}' \rangle + \nu \nabla^2 \bar{\boldsymbol{\omega}}_T + \nabla \times \langle \mathbf{b}' \cdot \nabla \mathbf{b}' \rangle + \\ & \bar{\boldsymbol{\omega}}_c \cdot \nabla \bar{\mathbf{v}} + \bar{\boldsymbol{\omega}} \cdot \nabla \bar{\mathbf{v}}_c + \bar{\boldsymbol{\omega}}_c \cdot \nabla \bar{\mathbf{v}}_c, \end{aligned} \quad (5)$$

where we have neglected terms second order in time-varying mean quantities. Subtracting (5) from (2) gives

$$\begin{aligned} d_t \boldsymbol{\omega}' = & \boldsymbol{\omega}' \cdot \nabla \bar{\mathbf{v}}_T - \bar{\mathbf{v}} \cdot \boldsymbol{\omega}' + \bar{\boldsymbol{\omega}}_T \cdot \nabla \mathbf{v}' - \mathbf{v}' \cdot \nabla \boldsymbol{\omega}' \\ & - \mathbf{v}' \cdot \nabla \bar{\boldsymbol{\omega}}_T + \boldsymbol{\omega}' \cdot \nabla \mathbf{v}' - \nabla \times \langle \mathbf{v}' \times \boldsymbol{\omega}' \rangle + \nu \nabla^2 \boldsymbol{\omega}' + \\ & \nabla \times (\mathbf{b}' \cdot \nabla \bar{\mathbf{b}}) + \nabla \times (\bar{\mathbf{b}} \cdot \nabla \mathbf{b}') + \nabla \times (\mathbf{b}' \cdot \nabla \mathbf{b}') - \\ & \nabla \times \langle \mathbf{b}' \cdot \nabla \mathbf{b}' \rangle + \nabla \times \mathbf{F}'(\mathbf{x}, t). \end{aligned} \quad (6)$$

Similarly, the equation for the mean B-field, derived by averaging (3) is

$$d_t \bar{\mathbf{b}} = \nabla \times \langle \mathbf{v}' \times \mathbf{b}' \rangle + \bar{\mathbf{b}} \cdot \nabla \bar{\mathbf{v}}_c + \nu_M \nabla^2 \bar{\mathbf{b}}. \quad (7)$$

Subtracting (7) from (3) yields the equation for the fluctuating B-field

$$\begin{aligned} d_t \mathbf{b}' = & \mathbf{b}' \cdot \nabla \bar{\mathbf{v}}_T - \bar{\mathbf{v}} \cdot \nabla \mathbf{b}' + \bar{\mathbf{b}} \cdot \nabla \mathbf{v}' - \mathbf{v}' \cdot \nabla \bar{\mathbf{b}} + \\ & \mathbf{b}' \cdot \nabla \mathbf{v}' - \mathbf{v}' \cdot \nabla \mathbf{b}' - \nabla \times \langle \mathbf{v}' \times \mathbf{b}' \rangle + \nu_M \nabla^2 \mathbf{b}'. \end{aligned} \quad (8)$$

Growth of  $\bar{\mathbf{b}}$  or  $\bar{\boldsymbol{\omega}}$  requires the  $\nabla \times$  terms in (7) and (5) to be non-vanishing. However, when the turbulence is strictly homogeneous in  $\mathbf{v}'$  and  $\mathbf{b}'$ , these quantities vanish. For purely isotropic turbulence, the term  $\langle \mathbf{v}' \times \mathbf{b}' \rangle$  in (7) vanishes since it is the average of a vector, while  $\nabla \times \langle \mathbf{v}' \times \boldsymbol{\omega}' \rangle \propto \nabla \times \nabla \langle v'^2 \rangle$  and  $\nabla \times \langle \mathbf{b}' \cdot \nabla \mathbf{b}' \rangle \propto \nabla \times \nabla \langle b'^2 \rangle$  vanish from Reynolds rules and incompressibility. Thus, anisotropy and inhomogeneity must be present for nontrivial time evolution of mean fields.

We expand the turbulent quantities on the right of Eqs. (5) and (7) to linear order in *both*  $\bar{\mathbf{b}}$  and  $\bar{\mathbf{v}}$  using the equations for the fluctuating fields thus generalizing the approach of (Field & Blackman, 1996) where only the mean B-field was used. To find the lowest order terms, we assume weakly anisotropic inhomogeneous turbulence: Terms linear in the time-varying mean quantities contribute, but their averaged  $0^{th}$  order coefficients, are taken to be isotropic and homogeneous (still allowing for reflection asymmetry). Iterating the equations using the formal solutions for the turbulent fields  $\mathbf{b}'(t) = \mathbf{b}'(t=0) +$

$\int d_t' \mathbf{b}' dt'$  and  $\boldsymbol{\omega}'(t) = \boldsymbol{\omega}'(t=0) + \int d_t' \boldsymbol{\omega}' dt'$ , and using times appropriately chosen such that the correlations  $\langle \mathbf{v}'(t) \times \boldsymbol{\omega}'(0) \rangle = \langle \mathbf{v}'(t) \times \mathbf{b}'(0) \rangle \simeq 0$ , we obtain to first order in mean quantities

$$\langle \mathbf{v}' \times \boldsymbol{\omega}' \rangle^{(1)} = \langle \mathbf{v}'^{(0)}(t) \times \int_0^t d_t' \boldsymbol{\omega}'^{(1)} dt' \rangle + \langle \int_0^t d_t' \mathbf{v}'^{(1)} dt' \times \boldsymbol{\omega}'^{(0)}(t) \rangle, \quad (9)$$

with similar expressions for  $\langle \mathbf{b}' \cdot \nabla \mathbf{b}' \rangle^{(1)}$  and  $\langle \mathbf{v}' \times \mathbf{b}' \rangle^{(1)}$ . The calculation of these averages requires Eqs. (4), (6), and (8) for the time integrands. Using (4) also requires an expression for the pressure, which arises in (9) and in  $\langle \mathbf{v}' \times \mathbf{b}' \rangle$  via the terms

$$\langle \boldsymbol{\omega}'^{(0)}(t) \times \int_0^t \nabla p'^{(1)} dt' \rangle \text{ and } \langle \mathbf{b}'^{(0)}(t) \times \int_0^t \nabla p'^{(1)} dt' \rangle. \quad (10)$$

Using isotropy, homogeneity, and Reynolds rules, we now show that remarkably, terms of the form (10) vanish in the derivation of the mean field equations to the order considered. Consider the pressure in the isoentropic energy equation (hereafter neglecting microphysical fluid and magnetic viscosities)  $D_t P \equiv d_t P + (\mathbf{v}' + \bar{\mathbf{v}}) \cdot \nabla P = -\gamma P \nabla \cdot \mathbf{v}$ , where  $\gamma$  is the adiabatic index (Batchelor 1967, Boyd and Sanderson 1969). We have assumed incompressibility ( $\nabla \cdot \mathbf{v} \simeq 0$  and constant  $\rho$ ) above, but here we must allow  $\nabla \cdot \mathbf{v} \neq 0$  for the last term in the energy equation: Even though  $\nabla \cdot \mathbf{v}$  is small, the sound speed is large (*i.e.*  $\gamma$  is large, see Batchelor 1967) such that the magnitude of  $DP/Dt$  cannot be ignored. This is why the energy equation is normally not useful in incompressible approaches. But here it will be symmetry properties of the energy equation that account for the absence of a pressure contribution when used in (10) and the seemingly offending term will not contribute. Taking  $p' \equiv P'/\rho$  we obtain

$$d_t p' = \langle \mathbf{v}' \cdot \nabla p' \rangle - \mathbf{v}' \cdot \nabla \bar{p} - \bar{\mathbf{v}} \cdot \nabla p' - \mathbf{v}' \cdot \nabla p' + \gamma \langle p' \nabla \cdot \mathbf{v}' \rangle - \gamma p' \nabla \cdot \bar{\mathbf{v}} - \gamma \bar{p} \nabla \cdot \mathbf{v}' - \gamma p' \nabla \cdot \mathbf{v}', \quad (11)$$

where  $\nabla \cdot \bar{\mathbf{v}}_c \simeq 0$ , but  $\nabla \cdot \bar{\mathbf{v}}, \nabla \cdot \mathbf{v}' \neq 0$ . The averaged terms in (11) make no contribution when inserted into (10) according to the Reynolds rules. We only consider terms *explicitly* linear in time-varying mean quantities, and iterate to first order in the correlation time; higher order terms would come from successive iterations in the equations for the fluctuating variables. This first order smoothing approximation (FOSA) (Krause & Rädler 1980) is “justified” when the correlation time of the turbulence is less than the eddy turnover time (Ruzmaikin et al. 1988, Blackman 1995), as is possibly the case in galaxies (Ruzmaikin et al. 1988), or if higher order correlations are otherwise reduced. Integrating (11), and using the assumptions discussed above,

$$\nabla p'^{(1)} = \nabla p'^{(1)}(0) - \nabla \int (\mathbf{v}'^{(0)} \cdot \nabla \bar{p} + \bar{\mathbf{v}} \cdot \nabla p'^{(0)} + \gamma p'^{(0)} \nabla \cdot \bar{\mathbf{v}} + \gamma \bar{p} \nabla \cdot \mathbf{v}'^{(0)}) dt'. \quad (12)$$

The pressure dependent contribution to  $\langle \mathbf{v}'^{(0)} \times \boldsymbol{\omega}'^{(0)} \rangle$  can then be written

$$\begin{aligned} \langle \boldsymbol{\omega}'^{(0)} \times \int_0^t d_t' \nabla p' dt' \rangle_k &= \epsilon_{ijk} \epsilon_{ims} \times \\ &\int_0^t dt' \int_0^{t'} dt'' \langle (\partial_j \bar{v}_l \partial_l p''^{(0)} + \bar{v}_l \partial_j \partial_l p''^{(0)} + \\ &\partial_j v_l^{(0)} \partial_l \bar{p} + v_l^{(0)} \partial_j \partial_l \bar{p} + \gamma p'^{(0)} \partial_j \partial_l \bar{v}_l + \\ &\gamma \partial_j p'^{(0)} \partial_l \bar{v}_l + \gamma \bar{p} \partial_j \partial_l v_l^{(0)} + \gamma \partial_j \bar{p} \partial_l v_l^{(0)}) \\ &\times (\partial_m v_s^{(0)} - \partial_s v_m^{(0)}) \rangle \\ &= \frac{1}{6} \int_0^t dt' \int_0^{t'} dt'' \langle \boldsymbol{\omega}'^{(0)}(t') \cdot \boldsymbol{\omega}'^{(0)}(t'') \rangle \partial_k \bar{p}(t), \end{aligned} \quad (13)$$

where we assume mean fields vary on time scales longer than the fluctuating fields, and  $\nabla p'^{(1)}(t=0)$  is uncorrelated with  $\mathbf{b}'(t)$  or  $\boldsymbol{\omega}'(t)$ . The only surviving term arises from the third term on the right hand side of (13) and the vanishing of the

remaining terms follows from careful application of isotropy (*i.e.* rank 2 and rank 3 averaged tensors of fluctuating  $0^{th}$  order quantities are proportional to  $\delta_{ij}$  and  $\epsilon_{ijk}$  respectively) and homogeneity (*i.e.*  $\partial_i \langle X_j X_k \rangle^{(0)} = 0$ ) of the  $0^{th}$  order turbulence, *without* invoking  $\nabla \cdot \mathbf{v}' = 0$ . The surviving term in (13) vanishes when put inside the curl in (5). A similar analysis holds for the second term in (10) when put into (7); thus, the pressure does not contribute to (5) or (7).

Approximating time integrals by factors of the correlation time  $\tau_c$  (Ruzmaikin et al. 1988), and freely employing Reynolds rules and incompressibility we obtain

$$\begin{aligned} (3/\tau_c) \nabla \times \langle \mathbf{v}' \times \boldsymbol{\omega}' \rangle^{(1)} &= \langle v_i'^{(0)} \omega_i'^{(0)} \rangle (\nabla \times \bar{\boldsymbol{\omega}}) + \\ &\langle v_i'^{(0)} v_i'^{(0)} \rangle \nabla^2 \bar{\boldsymbol{\omega}} + 2 \langle \omega_i'^{(0)} b_i'^{(0)} \rangle \nabla^2 \bar{\mathbf{b}} + \\ &\langle \epsilon_{ijk} \omega_k'^{(0)} \partial_i b_j'^{(0)} \rangle (\nabla \times \bar{\mathbf{b}}) - \langle v_i'^{(0)} b_i'^{(0)} \rangle \nabla^2 (\nabla \times \bar{\mathbf{b}}) \end{aligned} \quad (14)$$

with similar expressions for  $\nabla \times \langle \mathbf{b}' \cdot \nabla \mathbf{b}' \rangle^{(1)}$  and  $\nabla \times \langle \mathbf{v}' \times \mathbf{b}' \rangle^{(1)}$ . Upon substituting these into (5) and (7), the curls can be pulled onto the  $\bar{\boldsymbol{\omega}}$  and  $\bar{\mathbf{b}}$  from homogeneity of  $0^{th}$  order averages. Note that  $\bar{\boldsymbol{\omega}}_c$  and  $\bar{\mathbf{v}}_c$  contribute to the  $0^{th}$  order quantities, and thus do not show up explicitly in (13). After some simplification, we obtain

$$\begin{aligned} d_t \bar{\boldsymbol{\omega}} &= \alpha_0 (\nabla \times \bar{\boldsymbol{\omega}}) + \alpha_1 (\nabla \times \bar{\mathbf{b}}) + \beta_0 \nabla^2 \bar{\boldsymbol{\omega}} + \beta_1 \nabla^2 \bar{\mathbf{b}} \\ &- \alpha_2 \nabla^2 (\nabla \times \bar{\mathbf{b}}) + \bar{\boldsymbol{\omega}}_c \cdot \nabla \bar{\mathbf{v}} + \bar{\boldsymbol{\omega}} \cdot \nabla \bar{\mathbf{v}}_c + \bar{\boldsymbol{\omega}}_c \cdot \nabla \bar{\mathbf{v}}_c, \end{aligned} \quad (15)$$

and

$$d_t \bar{\mathbf{b}} = \alpha_2 (\nabla \times \bar{\boldsymbol{\omega}}) + \alpha_3 (\nabla \times \bar{\mathbf{b}}) + \beta_2 \nabla^2 \bar{\mathbf{b}} + \bar{\mathbf{b}} \cdot \nabla \bar{\mathbf{v}}_c \quad (16)$$

where the coefficients are:

$$\begin{aligned} \alpha_0 &= (\tau_c/3) \langle \boldsymbol{\omega}'^{(0)} \cdot \mathbf{v}'^{(0)} \rangle \\ \alpha_1 &= (\tau_c/3) \langle \boldsymbol{\omega}'^{(0)} \cdot \nabla \times \mathbf{b}'^{(0)} \rangle \\ \alpha_2 &= (2\tau_c/3) \langle \mathbf{b}'^{(0)} \cdot \mathbf{v}'^{(0)} \rangle \\ \alpha_3 &= (2\tau_c/3) \langle \mathbf{b}'^{(0)} \cdot \nabla \times \mathbf{b}'^{(0)} \rangle - \alpha_0 \\ \beta_0 &= (\tau_c/3) \langle \mathbf{v}'^{(0)} \cdot \mathbf{v}'^{(0)} + \mathbf{b}'^{(0)} \cdot \mathbf{b}'^{(0)} \rangle \\ \beta_1 &= (2\tau_c/3) \langle \boldsymbol{\omega}'^{(0)} \cdot \mathbf{b}'^{(0)} \rangle \\ \beta_2 &= (\tau_c/3) \langle \mathbf{v}'^{(0)} \cdot \mathbf{v}'^{(0)} + 2\mathbf{b}'^{(0)} \cdot \mathbf{b}'^{(0)} \rangle, \end{aligned} \quad (17)$$

### 3. Solutions and Discussion

The coefficients  $\alpha_3$  and  $\beta_2$  in (16) are modified helicity and diffusion coefficients of the standard magnetic dynamo theory. A similar, but not identical, set of helicity and diffusion coefficients enter (15) for the mean vorticity. The cross-correlations  $\alpha_1, \alpha_2, \beta_1$ , taken to vanish in Vainshtein & Kitchatinov 1983, need not vanish, as they preserve all MHD symmetries in (15) and (16). We will now see how the finite helicity is required for both B-field (Moffatt 1978) and vorticity growth (Kitchatinov et al. 1994a).

To show the growth, we assume constant dynamo coefficients and consider the simple case when the last 3 terms in (15) and the last term in (16) are negligible. This assumption is valid in systems for which  $\bar{\boldsymbol{\omega}}_c$  is perpendicular to the gradient of the time varying mean quantities, and for which the former is relatively uniform or perpendicular to the dominant B-field component. (The influence of  $\bar{\boldsymbol{\omega}}_c$  is always present however, in providing reflection asymmetry to the  $0^{th}$  order turbulent quantities.) The simplified equations yield a generalized vorticity-magnetic  $\alpha^2$  dynamo.

The mean-field equations are then solved by assuming solutions of the form  $\exp(\sigma t - i\vec{k} \cdot \vec{r})$  to obtain the linear algebraic equations  $\sigma \vec{X} = \mathbf{M} \vec{X}$ ,  $\vec{X} = (\bar{\omega}_x, \bar{\omega}_y, \bar{\omega}_z, \bar{b}_x, \bar{b}_y, \bar{b}_z)$ . The eigenvalues  $\sigma$  of the dynamical matrix  $\mathbf{M}(\vec{k})$  determine the time evolution of the fields at wavevector  $\vec{k}$ . Different behavior can be inferred from the general solution  $\vec{X}(\vec{r}, t) = \sum_{j=1,6} c_j \vec{\xi}_j \exp(\sigma_j t - i\vec{k} \cdot \vec{r})$ . Here,  $c_j$  are depend on the initial values  $X_j(t=0)$ , and  $\vec{\xi}_i$  are eigenvectors which are

linear combinations of  $\bar{\omega}$  and  $\bar{\mathbf{b}}$ . When cross-correlations between the turbulent velocities and B-fields are non-vanishing, the mean vorticity and B-fields are naturally coupled. The general  $6 \times 6$  matrix equation can be solved numerically for  $X_j$  and contains all decaying and growing modes. Because the  $c_j$  are combinations of the initial  $X_j(0)$ , for growing modes, all components of  $\bar{\mathbf{b}}$  and  $\bar{\omega}$  will in general grow unless very specific initial conditions  $\mathbf{X}(t=0)$  and values of  $k$  and  $\sigma(k)$  make certain  $c_j$  vanish.

First we consider thin disk systems which have only in-plane mean field components (while allowing the turbulence to be 3-D), *i.e.*,  $\bar{\mathbf{v}} = (\bar{v}_x, \bar{v}_y, 0)$ ,  $\bar{\mathbf{b}} = (\bar{b}_x, \bar{b}_y, 0)$ . The three possible poles are  $\sigma_0 = -\beta_0 k^2$  and  $2\sigma_{\pm} = -(\beta_0 + \beta_2)k^2 \pm |k|\sqrt{4\alpha_1\alpha_2 + (\beta_0 - \beta_2)^2 k^2 + 4\alpha_2^2 k^2}$ . which have growing modes ( $\sigma > 0$ ) when  $k^2 < \alpha_1\alpha_2/(\beta_0\beta_2 - \alpha_2^2)$  (if and only if  $\alpha_1\alpha_2/(\beta_0\beta_2 - \alpha_2^2) > 0$ ). Thus, the growth of  $\bar{\omega}_z$  and  $(\bar{b}_x, \bar{b}_y)$  depends on the signs and magnitudes of  $\alpha_1, \alpha_2, \beta_0$ , and  $\beta_2$ . because we allow the small scale

The fully 3-D case requires numerical calculation of  $\sigma_i$ , but we can obtain approximate behavior by truncating (15) and (16) to lowest order in mean field gradients. By taking the curl of (16) and using  $\bar{\mathbf{j}}$  we see that terms with one higher derivative in the mean quantities are reduced by a factor  $\ell/L$ . If  $\alpha_2 \propto \langle \mathbf{v}^{(0)} \cdot \mathbf{b}^{(0)} \rangle$  is negligible initially, it will remain so (Chandran 1997, Kraichnan 1959). Neglecting only the  $\alpha_2$  term in (15) and (16), the six roots are  $\sigma = -\beta_0 k^2, -\beta_2 k^2, -\beta_0 k^2 \pm \alpha_0 |k|, -\beta_2 k^2 \pm \alpha_3 |k|$ , which displays two growing modes for small  $k$  depending on the signs of  $\alpha_0$  and  $\alpha_3$  ( $\beta_0, \beta_3 > 0$ ).

Note that if cross-correlations between turbulent velocities and magnetic fields vanish, the equations for  $\bar{\omega}$  and  $\bar{\mathbf{b}}$  decouple into separate MFMD and MFVD equations of the form  $\partial_t \bar{\mathbf{m}} = \alpha \nabla \times \bar{\mathbf{m}} + \beta \nabla^2 \bar{\mathbf{m}}$  where  $\bar{\mathbf{m}} = \bar{\omega}$  or  $\bar{\mathbf{b}}$ , with their respective helicity and diffusion coefficients. The time evolution of the decoupled dynamos is determined by the eigenvalues  $\sigma = -\beta k^2, \pm \alpha |k| -$

$\beta k^2$ . Thus, there is a growing solution ( $\sigma = |\alpha||k| - \beta k^2 > 0$ ) for small wavevectors  $|k| < |\alpha|/\beta$ . The most unstable mode is  $k^* = |\alpha|/(2\beta)$  which grows at a rate  $\sigma(k^*) = |\alpha|^2/(4\beta)$ , like the usual  $\alpha^2$  dynamo (Moffatt 1978).

To conclude: We have generalized the mean-field dynamo to include mean vorticity and have considered solutions of an  $\alpha^2$  vorticity-magnetic dynamo. An important implication of (15) and (16) is that the mean B-field can grow even when its initial value is zero because a seed mean B-field can be generated by a coupling between the small scale B-field to the small scale velocity, and the latter to the mean vorticity. This feature should be preserved in more general treatments as it simply results from the coupling between (15) and (16) and only requires non-vanishing cross correlations. Growth requires helical turbulence which can be supplied by an underlying global rotation. Astrophysical applications of more extensive treatments may include accretion disks, where vortices and B-fields can combine to concentrate emerging radiation into strong beams (Acosta-Pulido et al. 1990, Abramowicz 1992). Also, Yoshizawa & Yokoi (1993) showed that an equation similar to (16) can lead to generation of toroidal fields whose upward pressure drives a jet that subsequently may develop poloidal fields. The CMFD may provide the sustaining feedback.

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